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Series study of the continuous-spin Ising model

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Abstract. We study the critical exponents for the susceptibility and the correlation length for three continuous-spin models, which we call the border model, the double-well model and the single-well model, by the method of high-temperature series. We observe variations from the predictions of universality, but in some cases longer series may be required before firm conclusions can be drawn.

Wang and Baker [1] have recently aroused new interest in the behaviour of the critical indices as a function of the parameter in the continuous-spin Ising model. They studied this problem using a Monte Carlo method. We seek to examine it by series techniques. There have been previous studies of these sorts of questions by Barma and Fisher [2] and Baker and Johnson [3], but our study differs from these in the approaches used and the models studied.

In our selection of which models to study, we have kept in mind the ideas of Baker [4] who pointed out that there is reason to believe that the five cases—the Gaussian model, the single-well model, the border model, the double-well model and the Ising model—have values of the correlation length index ν which differ from that of neighbouring models. The current interest in this study relates to whether or not universality holds and if it does not, can the different models be assigned to the categories predicted by conformal invariance. We find, with the possible exception of the single-well case, where our results are least well converged, that there is a reasonably significant difference in at least one of the critical indices between adjacent models, in accord with the aforementioned ideas and that it is possible to identify at least one of the allowed conformally invariant sets of critical indices with our estimates of the critical indices for each of the models.

Specifically, the models we consider are defined by the partition function

$$Z = M^{-1}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{i=1}^{N}\mathrm{d}s_i\exp\left[\sum_i\left(K\sum_{\{\delta\}}s_is_{i+\delta}-\tilde{g}_0s_i^4-\tilde{A}s_i^2+Hs_i\right)\right] \qquad (1)$$

where M is a formal normalization constant such that $Z(H=K=0)=1$, N is the number of lattice sites, $\{\delta\}$ is one half the set of nearest-neighbour lattice vectors, and $K=J/kT$

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with J the exchange integral, k Boltzmann's constant and T the absolute temperature. A normalization of the spin variable is imposed by the condition that

$$1 = \langle s^2 \rangle = \frac{\int_{-\infty}^{+\infty} s^2 \exp(-\tilde{g}_0 s^4 - \tilde{A}s^2)\,\mathrm{d}s}{\int_{-\infty}^{+\infty} \exp(-\tilde{g}_0 s^4 - \tilde{A}s^2)\,\mathrm{d}s} \,. \tag{2}$$

By this condition we define $\tilde{A}$ as a function of $\tilde{g}_0$ such that $\tilde{A}(\tilde{g}_0)$ is analytic for $0 < \tilde{g}_0 < \infty$ and $\tilde{A}(0) = \frac{1}{2}$, $\lim_{\tilde{g}_0 \to \infty} \tilde{A}_0/\tilde{g}_0 = -2$. The crossing $\tilde{A}(\tilde{g}_b) = 0$ occurs for $\tilde{g}_b = [\Gamma(\frac{3}{4})/\Gamma(\frac{1}{4})]^2$ and is called the border model. Any value of $\tilde{g}_0$ such that $0 < \tilde{g}_s < \tilde{g}_b$ corresponds to a single-peaked potential in (1). We select for our single-well model $\tilde{g}_s = 0.037\,885$ to facilitate comparison with the results of Wang and Baker. Any value of $\tilde{g}_0$ such the $\tilde{g}_b < \tilde{g}_d < \infty$ corresponds to a two-peaked potential in (1). We select for our double-well model $\tilde{g}_d = 0.467\,251$, again to facilitate comparison with the results of Wang and Baker. We will study the plane square lattice, when the lattice is not mentioned, and will also give some results for the triangular lattice. Similar calculations have also been performed for the Ising model as a check of our method and comparable results have been observed.

First we discuss our analysis of the border model†. We have first studied the correlation length series by the method of Dlog Padé approximants [6]. The results for the near-diagonal approximants which use the series terms from order 14 onward give a rather stable picture and imply that $K_c \simeq 0.3285 \pm 1$ and $\nu \simeq 1.024 \pm 2$, where the errors are quoted in the last place given. This result is well within the ranges quoted by Baker and Johnson [3] and Wang and Baker [1]. We have checked the results by the method of integral approximants [6] and find agreement. The same checks were performed on the other calculations which we report below, but we will not say so in each case. Next we have studied the susceptibility. We find that the critical point here has a complex structure. We have used the method of integral approximants obtained from a second-order differential equation as explained in [7] to analyse this series for a confluent singularity. We find, when we specify the critical point as determined above, that $\gamma = 1.78 \pm 0.02$, but the character of the sub-dominant singularity if any is not resolved. One possible interpretation of our results is that there might be a strong singularity for K a little larger than K_c corresponding to a value of $\gamma_2 = 1.85$–2.16.

If we use biased Dlog Padé approximants to estimate the susceptibility critical exponent γ, we find $\gamma \simeq 1.79 \pm 1$, with stability from about the 12th term onwards. The result is to be compared with the Monte Carlo results of Wang and Baker of $\gamma \simeq 1.78 \pm 1$. It is less than that of Baker and Johnson $\gamma = 2.00$ derived with much shorter series. We have also analysed the results on the triangular lattice for the 10-term border-model series. Here the susceptibility series is better behaved than the correlation-length series. We agree with reference [3] for $K_c = 0.213 \pm 1$, $\gamma = 1.91 \pm 2$, where the apparent error estimates for γ should realistically be increased to around ± 0.1 by cross comparison with other methods of analysis. We add the analysis of $\nu = 1.06 \pm 2$, which is consistent within error with the plane square-lattice results. The method of conformal invariance predict a table of possible values for γ and ν [8]. The possible value are further limited by modular invariance [9]. In this table, for the allowed value of c, the central charge, which is the key parameter of conformally invariant theory,

$$c = 1 - \frac{6}{m(m+1)} \qquad 2 \leqslant m \leqslant \infty \tag{3}$$

† We wish to thank B Nickel [5] for making available to us his unpublished 21-term series in K for the susceptibility and the square of the correlation length for this model.

corresponding to $m = 12$ ($\Delta_1 = (6,5), \Delta_2 = (7,7)$) we have $\gamma = 1.7628$, $\nu = 1.0353$, which are reasonably close to our results. We point out that for this selection the value of $c(2-\alpha)^2$ is closer to 4 than to the value of 2 shown by Wang and Baker [1] (α is the specific-heat critical exponent). We point out that there is some uncertainty in interpreting their results on this point because their expression involves a factor $(K_c - K)^2$ and they only work with one size lattice (128×128). For the Ising model, as they point out, a finite-size shift of the order of 0.3% in K_c is expected. Since they plot $c(2-\alpha)^2$ to within 1% of the critical temperature and then extrapolate, a factor of two in the result is conceivable.

For the double-well model, we have used the series due to Baker and Kincaid [10]. This is a 10-term series and the reader is cautioned that the quality of the results may suffer from 'short-series' effects. In this case we started with an analysis of the susceptibility series by the Dlog Padé method. We found $K_c \simeq 0.3737 \pm 3$, with stability found over approximants which involve the 9- and 10-term series. This value of K_c lies within the error range quoted by Wang and Baker. We find $\gamma \simeq 1.90 \pm 2$. If we bias the Dlog Padé method to this value of K_c we find $\nu \simeq 1.08 \pm 1$. For the triangular lattice we get $K_c = 0.238 \pm 1$, $\gamma = 1.85$–1.90 and $\nu = 1.07 \pm 1$. The value of γ agrees with that of Wang and Baker within their errors. We find for conformal invariant theories that the case for $m = 9$ ($\Delta_1 = (5,4)$, $\Delta_2 = (5,5)$) gives the values $\gamma = 1.900\,00$, $\nu = 1.083\,33$, which agree with our results. Here the values of $c(2-\alpha)^2 \simeq 2.0833$ are well within the range of values tabulated by Wang and Baker.

Finally we have studied the single-well model by the same methods as noted above. Again we use the 10-term series of Baker and Kincaid. Analysis of the χ series implies $K_c \simeq 0.298 \pm 1$ and the concomitant value of $\gamma \simeq 1.735 \pm 30$. The value of K_c is in agreement with that of Wang and Baker, but they quote a lower value $\gamma = 1.64 \pm 1$ for this case. Using our value of K_c we find $\nu \simeq 0.93 \pm 2$. For the triangular lattice we get $K_c = 0.195 \pm 2$, $\gamma = 1.7$–1.9 (erratic) and $\nu = 0.93 \pm 1$. For the value of c corresponding to $m = 6$ ($\Delta_1 = (3,2)$, $\Delta_2 = (3,3)$) we have the values $\gamma = 1.714\,28$ and $\nu = 0.952\,38$. This selection corresponds to a value of $c(2-\alpha)^2$ of about 3, and we repeat the same remarks here that we made on this point concerning the border model.

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